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Thermodynamics of photons in relativistic $e^+e^-\gamma$ plasmas

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Thermodynamic and spectral properties of a photon gas in $e^+e^-\gamma$ plasmas are studied. The effect of a finite effective mass of a photon, associated with the plasma frequency cutoff, is self-consistently included. In the ultrarelativistic plasma, the photon spectrum turns out to be universal with the temperature normalized plasma frequency cutoff as a fundamental constant independent of plasma parameters. Such a universality does not hold in the nonrelativistic plasma. [S1063-651X(99)50305-X]

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It is well known that a photon gas, which is in thermodynamic equilibrium with matter, exhibits the Planckian blackbody energy spectrum. This is correct only if matter is electrically neutral (for example, an atomic gas) where the thermodynamic equilibrium is established via photon emission and absorption in atomic transitions. Such an equilibrium no longer forms in ionized gases (plasmas) where the number of charged particles is not negligible. An electromagnetic wave (photon) is strongly coupled to collective plasma excitations via oscillating electric and magnetic fields. The dispersion relation of photons in a plasma is $\omega^2 \sim \omega_p^2 + k^2c^2$, where ω_p is referred to as the plasma frequency, which depends on plasma parameters (e.g., densities of electrons and positrons). Quite interestingly, such a dispersion may be interpreted as a manifestation of a finite *effective mass* of a photon, $m_\gamma \sim \hbar\omega_p/c^2$. There are no photons with energies lower than its effective rest energy. Hence, one may expect a cutoff in the Planckian spectrum for frequencies $\omega < \omega_p$. A new, modified blackbody distribution of photons forms in this case. The shape of the spectrum is controlled by the densities of electrons and positrons. It is remarkable that in the relativistic plasma, the electron and positron number densities are, in turn, *self-consistently* determined from the *equilibrium* between the photons and the electrons, and positrons, with respect to the e^+e^--pair production.

Objects with hot, positron-electron-photon ($e^+e^-\gamma$) plasmas, referred to as fireballs, are not rare in astrophysics. For example, such fireball conditions are believed to be realized at early stages of the formation of the universe (see, for example, [1]), in energetic cosmological explosions that produce γ-ray bursts (see [2] for review), etc. Hence, the study of thermodynamic and statistical properties of a gas of photons in the $e^+e^-\gamma$ fireballs is of both academic and practical interest.

In this paper, we find a self-consistent, blackbody radiation spectrum and thermodynamic properties of the photon gas in the $e^+e^-\gamma$ fireball plasma. In particular, the equilibrium spectrum of photons in the ultrarelativistic case ($T \gg m_ec^2$) turns out to be *universal*:

(1) The frequency cutoff normalized by temperature is a *fundamental constant* independent of any plasma parameters.
(2) The simple, power-law scalings of thermodynamic parameters such as pressure, density, etc. vs temperature, valid for the Planck distribution, hold for ''massive'' photons as well. Numerical prefactors, however, modify self-consistently.

At the end of the paper, we also briefly consider a nonrelativistic case and discuss implications for the cosmic microwave background spectrum distortion, purely radiative fireball model of γ-ray bursts, and stellar interiors.

Here we should comment on one important point. The gas of photons obeys the Bose-Einstein statistics with a vanishing chemical potential and, also, is assumed to be in thermodynamic equilibrium with matter which, in general, has a nonvanishing chemical potential. On the other hand, one of the conditions of thermodynamic equilibrium is the equality of the chemical potentials of the interacting systems. This contradiction may be resolved by treating the process of radiation as a first-order phase transition (Bose condensation) and properly taking into account the equality of the chemical potentials of matter and radiation. An excellent discussion of this subject may be found in Ref. [3]. The deviation of a photon spectrum from the Planck distribution may be significant in the nonrelativistic plasmas, where thermodynamic equilibrium is established via, for example, Compton scattering of a photon on e^+ or e^-. In the ultrarelativistic regime (considered in this paper) two- and three-photon annihilation/creation of an e^+e^- pair dominate and are responsible for establishing thermodynamic equilibrium. The chemical potential of an e^+e^- pair is identically zero (by definition, e^+ and e^- have equal but opposite sign chemical potentials) and the condition of the equality of the chemical potentials of matter and radiation is satisfied automatically. The Planck statistics of photons is thus valid in the ultrarelativistic $e^+e^-\gamma$ plasma.

Ultrarelativistic case. The dispersion relation of electromagnetic waves in the ultrarelativistic plasma reads [4]

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$$\frac{k^2c^2}{\omega^2}=1+\frac{3}{4}\frac{\omega_{p,\mathrm{rel}}^2}{\omega kc}\left[\left(1-\frac{\omega^2}{k^2c^2}\right)\ln\left|\frac{\omega-kc}{\omega+kc}\right|-\frac{2\omega}{kc}\right], \quad (1)$$

where $\omega_{p,\mathrm{rel}}^2=\Sigma_\alpha 4\pi e_\alpha^2 n_\alpha c^2/3T$ is the relativistic plasma frequency, e_α and n_α are the charge and number density of species α, the sum Σ_α goes over all relativistic ($T\gg m_\alpha c^2$) species, i.e., electrons and positrons in our case, and the contribution from nonrelativistic species, which is mc^2/T times smaller, is neglected here. An exact analytical solution to this equation is unknown. The asymptotic behavior

$$\omega^2=\begin{cases}\omega_{p,\mathrm{rel}}^2+\dfrac{5}{6}k^2c^2, & \omega\gg kc\\[2mm] \dfrac{3}{2}\omega_{p,\mathrm{rel}}^2+k^2c^2, & \omega\to kc,\end{cases} \quad (2)$$

suggests, however, the following approximate solution to Eq. (1):

$$\omega^2=\omega_{p,\mathrm{rel}}^2+k^2c^2, \quad (1')$$

which roughly satisfies both asymptotes and, also, is the *exact* dispersion relation in the nonrelativistic plasma. A direct numerical comparison of Eqs. (1) and (1′) shows that the fractional error of the approximation does not exceed 3%.

In the ultrarelativistic plasma, the electrons and positrons are in thermal equilibrium. Hence, their number densities are equal and $n_{e^+}=n_{e^-}=\frac{7}{8}n_\gamma$, where the factor 7/8 accounts for the difference in Fermi and Bose statistics [5,6], and n_γ is the number density of photons. The relativistic plasma frequency now becomes

$$\omega_{p,\mathrm{rel}}^2=\sum_{\pm}\frac{4\pi e^2 n_\pm c^2}{3T}=\frac{7\pi e^2 n_\gamma c^2}{3T}. \quad (3)$$

Note here that the plasma frequency depends on the density of photons to be determined [7].

The occupation numbers for a gas of photons are given by the Bose-Einstein distribution with a vanishing chemical potential: $n(k)=\{\exp[\hbar\omega(k)/T]-1\}^{-1}$. The number density of photons is formally written as follows [5]:

$$n_\gamma=\frac{T^3}{\pi^2\hbar^3c^3}\int_0^\infty\frac{x^2dx}{\exp(\sqrt{7\pi e^2\hbar^2c^2 n_\gamma/3T^3+x^2})-1}, \quad (4)$$

where we made use of the fact that the degeneracy $g=2$ for photons (two independent polarizations). This equation *self-consistently* determines the number density of photons in the $e^+e^-\gamma$ fireball plasma where the plasma frequency cutoff (i.e., the effective photon mass), in turn, depends on n_γ.

To proceed further we note a remarkable fact. Namely, n_γ and T enter Eq. (4) in combination n_γ/T^3. Then, the solution to this equation must be sought in the form $n_\gamma=\mathrm{const}\times T^3$ with the const to be determined from Eq. (4). Thus, the parametrization $n_\gamma\propto T^3$ removes *explicit* temperature dependence of the integral in Eq. (4). Hence this solution for n_γ is *universal*. Note, this is usually not the case for particles with nonzero mass. Since $n(\mathbf{p})=[\exp(\sqrt{m^2c^4+|\mathbf{p}|^2c^2}/T)-1]^{-1}$, temperature enters the integral over particle momenta, $\mathbf{p}$, via the combination $(mc^2/T)^2$, so that all the thermodynamic quantities (e.g., density, energy, entropy, etc.) do not exhibit universal, power-law scalings. For photons, however, $m_\gamma\propto T$ and the universal (Planckian) scalings are recovered. Numerical coefficients, however, modify self-consistently.

Now, we write $n_\gamma=fn_{\mathrm{Pl}}$, where

$$n_{\mathrm{Pl}}=[2\zeta(3)/\pi^2](T/\hbar c)^3,$$

$\zeta(3)\simeq 1.2$ is the Riemann zeta function of 3, and f is a constant to be calculated. Hereafter, the subscript "Pl" denotes values calculated for the Planckian distribution. Then, Eq. (4) becomes

$$2\zeta(3)f=\int_0^\infty\frac{x^2dx}{\exp(\sqrt{\Delta^2f+x^2})-1}, \quad (5)$$

where

$$\Delta=\frac{\hbar\omega_{p,\mathrm{rel}}}{T}=\left(\alpha_e\frac{14\zeta(3)}{3\pi^2}\right)^{1/2}\simeq 6.44\times10^{-2} \quad (6)$$

and $\alpha_e=e^2/\hbar c$ is the fine structure constant. Given f, the energy density, pressure of the gas of photons, as well as other thermodynamic parameters, may be calculated straightforwardly [5]:

$$\epsilon_\gamma=\frac{T^4}{\pi^2\hbar^3c^3}\int_0^\infty\frac{\sqrt{\Delta^2f+x^2}\,x^2dx}{\exp(\sqrt{\Delta^2f+x^2})-1}, \quad (7a)$$

$$p_\gamma=\frac{1}{3}\frac{T^4}{\pi^2\hbar^3c^3}\int_0^\infty\frac{x^3dx}{\exp(\sqrt{\Delta^2f+x^2})-1}. \quad (7b)$$

Numerically solving Eq. (5) for $f\equiv(1+\delta f_n)$ and using Eqs. (7), one finally obtains

$$n_\gamma=(1+\delta f_n)n_{\mathrm{Pl}}, \quad \delta f_n\simeq 3.4\times10^{-3}, \quad (8a)$$

$$\epsilon_\gamma=(1+\delta f_\epsilon)\epsilon_{\mathrm{Pl}}, \quad \delta f_\epsilon\simeq 5.2\times10^{-4}, \quad (8b)$$

$$p_\gamma=(1+\delta f_p)p_{\mathrm{Pl}}, \quad \delta f_p\simeq 1.0\times10^{-3}, \quad (8c)$$

where $\epsilon_{\mathrm{Pl}}=3p_{\mathrm{Pl}}=(\pi^2/15)(T^4/\hbar^3c^3)$. It is well known that the equation of state of a gas of nonrelativistic massive particles, $T\ll mc^2$, is $p_{nr}=\frac{2}{3}\epsilon_{nr}$, and that of ultrarelativistic (or massless) particles, $T\gg mc^2$, is $p_r=\frac{1}{3}\epsilon_r$. Since photons in the relativistic plasma turn out to be "subrelativistic," $\hbar\omega_{p,\mathrm{rel}}/T\sim 5\times10^{-2}$, we expect that the effective equation of state of photons will be $p_\gamma=a\epsilon_\gamma$, $\frac{1}{3}<a<\frac{2}{3}$. Indeed, from Eqs. (8) it follows that $p_\gamma/\epsilon_\gamma\simeq(1+\delta f_p-\delta f_\epsilon)p_{\mathrm{Pl}}/\epsilon_{\mathrm{Pl}}$, and we have

$$p_\gamma\simeq\frac{1}{3}(1+5.0\times10^{-4})\epsilon_\gamma. \quad (9)$$

The blackbody equilibrium radiation spectrum from the ultrarelativistic $e^+e^-\gamma$ fireball plasma is easily obtained from Eq. (7a):

$$d\epsilon_\gamma(\omega_*)=\frac{T^4}{\pi^2\hbar^3c^3}\frac{\omega_*^2\sqrt{\omega_*^2-\Delta^2f}}{e^{\omega_*}-1}d\omega_*, \quad (10)$$

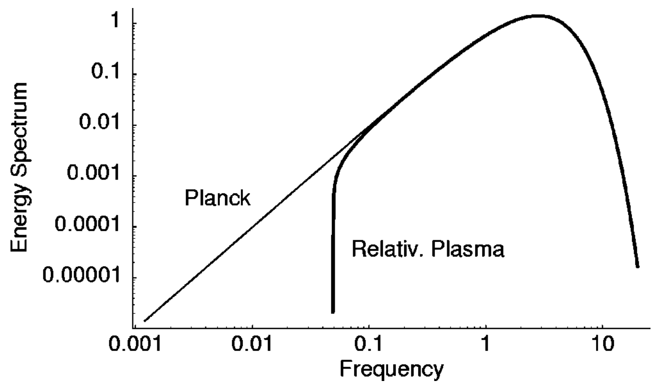

FIG. 1. Normalized blackbody equilibrium spectrum of radiation from the $e^+e^-\gamma$ relativistic plasma and the Planckian spectrum vs dimensionless frequency, $\omega_* = \hbar\omega/T$.

where $\omega_* = \hbar\omega/T$ is the dimensionless frequency. Note again that this spectrum is *universal*. The low-frequency cutoff scales with temperature in such a manner that $\hbar\omega_{p,\mathrm{rel}}/T = \Delta\sqrt{f} = \mathrm{const}$. The value of the low-frequency cutoff is a *fundamental constant* given by Eq. (6) [correction due to f, Eq. (8a) is small and may be neglected] and independent of any parameters of the plasma. The universal, blackbody energy spectrum of radiation from the $e^+e^-\gamma$ relativistic plasma is shown in Fig. 1. The Planckian spectrum is shown for comparison.

At last we comment that the radiation from a relativistic fireball is *always blackbody*. Indeed, for a nonthermal radiation spectrum to occur, the linear size of the system should be $R < \tau/\sigma_T n_{\mathrm{Pl}}(T \sim m_e c^2) \lesssim 10^{-4}$ cm (σ_T is the Thompson cross section and $\tau \lesssim 1$ is the optical depth of the plasma), which is microscopically small.

Nonrelativistic case. The universality of the photon spectrum breaks down at plasma temperatures below $T \sim m_e c^2$. The dispersion relation is exactly Eq. (1′) with $\omega^2_{p,\mathrm{rel}}$ replaced by its nonrelativistic counterpart:

$$\omega_p^2 = \frac{4\pi e^2 n_{e^\pm}}{m_e}, \tag{11}$$

where $n_{e^\pm} \equiv n_{e^+} + n_{e^-}$ and contributions from heavier particles are neglected. When the temperature drops below $m_e c^2$ the number density of e^+e^- pairs decreases exponentially due to annihilation. Hence, we expect the nonrelativistic cutoff frequency to be smaller than that in the ultrarelativistic regime. The density of charged particles is easily calculated [5]:

$$n_{e^\pm} = \left[n_0^2 + \frac{3}{\pi^3} \left(\frac{m_e c}{\hbar} \right)^6 \left(\frac{T}{m_e c^2} \right)^3 \exp\left(-\frac{2 m_e c^2}{T} \right) \right]^{-1/2}, \tag{12}$$

where n_0 is the residual density of electrons at $T \to 0$ (i.e., in the absence of pair creation).

A few estimates below show that the effect of the plasma frequency cutoff is usually small, unless a plasma is extremely dense and ''cold.'' First, most of the explosion energy in cosmological, purely radiative fireballs (if any),—the γ-ray bursters (GRB)—is emitted when the system becomes optically thin for radiation. This occurs at $T \simeq 20$ keV [2] when the e^+e^--pair density drops to $n_{e^\pm} \sim 1/\sigma_T R_{\mathrm{thin}} \sim 10^{14}\,\mathrm{cm}^{-3}$. The low-frequency cutoff is $(\hbar\omega_p/T)_{\mathrm{GRB}} \sim 10^{-8}$, which is hardly observable. Second, distortions of the cosmic microwave background (CMB) energy spectrum may be traced up to redshifts of $z \sim 10^8$ [1]. Taking the residual density of electrons $n_0 \sim 10^{-9} n_{\mathrm{Pl}}$ and temperature $T \sim$ few keV, we estimate the cutoff $(\hbar\omega_p/T)_{\mathrm{CMB}} \sim 10^{-7}$. Third, in the center of the sun, $n_0 \sim 10^{26}\,\mathrm{cm}^{-3}$ and $T \sim 1.4$ keV, which yield the cutoff $(\hbar\omega_p/T)_{\mathrm{sun}} \sim 0.2$; that is, the cutoff is close to the Planckian thermal peak. The effect on the solar (and stellar) models will be addressed in future publications.

To conclude, the thermodynamics of a gas of photons in the e^+e^- plasma is studied. The deviation from the Planck distribution occurs because no photons with frequencies below the plasma frequency exist in the plasma. Equivalently, photons in the plasma acquire an effective mass. In the ultrarelativistic plasma, $T \gg m_e c^2$, the effective photon mass scales as $m_\gamma = T\Delta/c^2$ with Δ as the fundamental constant (independent of any plasma parameters) given by Eq. (6). Hence, the equilibrium (blackbody) spectrum exhibits universal properties. The power-law scalings of thermodynamic parameters vs temperature for massive photons are the same as those for the Planck distribution, but with different numerical prefactors. In contrast, thermodynamics of photons in the nonrelativistic plasma is not universal.

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[7] If there are other ultrarelativistic (i.e., effectively massless), charged species, then $n_\pm = g'_* g_\gamma n_\gamma$ in Eq. (3), where $g_\gamma = 2$, $g'_* = \Sigma'_{i=\mathrm{bosons}} g_i + \frac{7}{8}\Sigma'_{i=\mathrm{fermions}} g_i$, and ''prime'' denotes summation over charged, ultrarelativistic species only; see Ref. [6].